

A LINEAR BOUND ON THE GENERA OF HEEGAARD SPLITTINGS WITH DISTANCES GREATER THAN TWO

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Let M be a closed, orientable 3-manifold that admits a triangulation with t tetrahedra. Let Σ be a Heegaard surface for M . S. Schleimer [19, Theorem 11.1] showed that if $g(\Sigma) \geq 2^{2^{16}t^2}$, then the Hempel distance of Σ (denoted by $d(\Sigma)$, see Definition 6) is at most two. In this paper we improve this result in two ways: first, we obtain a linear bound. Second, we allow M to be any 3-manifold that admits a *generalized triangulation*, that is, a decomposition into *generalized tetrahedra*: tetrahedra with some vertices removed or truncated. See Definitions 4. We note that if there exists a compact 3-manifold N so that M is obtained from N by removing a (possibly empty) closed subsurface of ∂N , then M admits a generalized triangulation, see Lemma 5. Our main result is:

Theorem 1. *Let M be an orientable 3-manifold that admits a generalized triangulation with t generalized tetrahedra. Let Σ be a Heegaard surface for M .*

If $g(\Sigma) \geq 76t + 26$, then $d(\Sigma) \leq 2$.

Remarks. (1) Saul Schleimer remarks that in his dissertation he obtained a quadratic bound giving distance 3 and a linear bound giving distance 4. A corrected version of Schleimer's dissertation is available at [18].

(2) In [19] Schleimer showed that [19, Theorem 11.1], together with the generalized Waldhausen Conjecture, imply that any 3-manifold admits only finitely many Heegaard splitting of distance 3 or more. Since the publication of [19], T. Li [11] proved the generalized Waldhausen Conjecture, establishing this fact.

Outline. In Section 1 we explain our perspective of this work and list six open questions. In Section 2 we explain a few preliminaries. The work begins in Section 3, where we take a strongly irreducible Heegaard surface of genus at least $76t + 26$, color it, and analyze the coloring; the climax of Section 3 is Proposition 14, where we prove existence of a pair of pants with certain useful properties. In Section 4 we prove Theorem 1.

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1. OPEN QUESTIONS

Theorem 1 is a constraint on the distance of surfaces of genus $76t + 26$ or more. There are other constraints on the distance known, and by far the most important is A. Casson and C. McA. Gordon's theorem [1] that says that no Heegaard surface of an irreducible, non-Haken 3-manifold has distance one. Other examples include Haken's theorem that says that any Heegaard surface of a reducible 3-manifold has distance zero, and Li's theorem [10] that says that an irreducible, non-Haken 3-manifold admits only finitely many Heegaard surfaces of positive distance.

On the positive side, Casson and Gordon construct 3-manifolds admitting infinitely many strongly irreducible Heegaard surfaces, of unbounded genera. By [19] or Theorem 1 above, all but finitely many have distance exactly two. Hempel [4] shows that for any $g \geq 2$, there exists a sequence of 3-manifolds M_n and Heegaard splittings Σ_n for M_n , so that $g(\Sigma_n) = g$ and $\lim_{n \rightarrow \infty} d(\Sigma_n) = \infty$. T. Evans [2] improved this by constructing, given $g \geq 2$ and $d \geq 0$, a Heegaard splitting of genus g with distance at least d .

However, narrowing the gap between the constraints and the constructions seems difficult. For example, the answers to the following questions are not known:

- Questions 2.** (1) Given $g \geq 2$ and $d > 0$, does there exist a Heegaard surface Σ , so that $g(\Sigma) = g$ and $d(\Sigma) = d$?
- (2) Given $d > 0$, does there exist a Heegaard surface Σ , so that $d(\Sigma) = d$?
- (3) Given $g_i \geq 2$ and $d_i > 0$ ($i = 1, 2$), does there exist a 3-manifold admitting distinct Heegaard surfaces Σ_1, Σ_2 , so that $g(\Sigma_i) = g_i$ and $d(\Sigma_i) = d_i$?
- (4) Given $d_i > 0$ ($i = 1, 2$), does there exist a 3-manifold admitting distinct Heegaard surfaces Σ_1, Σ_2 , so that $d(\Sigma_i) = d_i$?

We comment on these questions. First, the word “distinct” can be interpreted as “distinct up to isotopy” or “distinct up to homeomorphism”. Both yield interesting questions. Next we comment that for certain values of g, d, g_1, g_2, d_1 , and d_2 there are known examples of 3-manifolds and surfaces that answer Questions 2 (1), (2), and (3) affirmatively; we will not attempt to list them here. On the other hand, there are constraints on the distance, notably [17, Corollary 3.5] where M. Scharlemann and M. Tomova prove that if Σ_1 and Σ_2 are distinct surfaces (up to isotopy) so that $d(\Sigma_2) > 2g(\Sigma_1)$, then $g(\Sigma_1) > g(\Sigma_2)$ and $d(\Sigma_1) = 0$.

The answer for Question 2 (4) is known only in the following three cases:

- $d_1 = d_2 = 2$: As mentioned above, there are examples of Casson and Gordon of 3-manifolds admitting infinitely many Heegaard surfaces of unbounded genera and of distance 2. Other examples follow from K. Morimoto and M. Sakuma [12]. They show that there exist 2-bridge knots K admitting more than one minimal genus Heegaard surfaces (up to homeomorphism). Let Σ be one of these surfaces. We show that $d(\Sigma) = 2$. First, since $g(\Sigma) = 2$, it is easy to see that $d(\Sigma) \geq 2$. Next, Σ is constructed by viewing K as a torus 1-bridge knot (*i.e.*, decomposing S^3 into two solid tori T_i , $i = 1, 2$, so that K intersects each T_i in a single unknotted arc) and tubing once. Meridian disks for T_i which

are disjoint from K and the tube, are also disjoint from the core of the tube, showing that $d(\Sigma) \leq 2$.

- $d_1 = d_2 = 1$: Let M be a 4-punctured sphere cross S^1 . J. Schultens [20] showed that $g(M) = 3$, and every minimal genus Heegaard surface is vertical. This implies that M admits two minimal genus Heegaard splittings, say Σ_1 and Σ_2 , such that Σ_1 is obtained by tubing three boundary parallel tori, and Σ_2 is obtained by tubing two boundary parallel tori, with an extra tube that wraps around a third boundary component. Since Σ_1 and Σ_2 induce boundary partitions with distinct numbers of components, they are distinct up to homeomorphism. By construction, $d(\Sigma_1) = d(\Sigma_2) = 1$.
- $d_1 = 1, d_2 = 2$: In [7] the authors constructed a 3-manifold M admitting Heegaard splittings Σ_1, Σ_2 , with $d(\Sigma_2) = 2$ and $d(\Sigma_1) = 1$. In this example, $g(M) = g(\Sigma_1) = g(\Sigma_2) = 3$.

We see that much is known when $d_1, d_2 \leq 2$. By contrast, the answers to the following basic questions are unknown:

- Questions 3.** (1) Does there exist a 3-manifold admitting two distinct Heegaard surfaces with distance 3 or more?
- (2) Does there exist a 3-manifold admitting a Heegaard surface of distance 3 or more that is not minimal genus?

2. PRELIMINARIES

By *manifold* we mean orientable 3-manifold. We assume familiarity with the basic notions of 3-manifold topology (see, for example, [5] or [6]) and the basic facts about Heegaard splittings (see, for example, [16] or [15]). We define:

- Definitions 4.** (1) Let T be a tetrahedron. A *generalized tetrahedron* is obtained by fixing V_1, V_2 , disjoint sets of vertices of T , removing V_1 and truncating V_2 ; that is, a generalized tetrahedron T' has the form $T' = \text{cl}(T \setminus N(V_2)) \setminus V_1$. T' has exactly four faces (resp. exactly six edges, at most four vertices), which are the intersection of the faces (resp. edges, vertices) of T with T' . In particular, no component of $N(V_2) \cap T'$ is considered a face. Important special cases are when $V_2 = \emptyset$, then T' is called *semi-ideal*, and when V_1 consists of all four vertices, then T' is called *ideal*.
- (2) A *generalized triangulation* is obtained by gluing together finitely many generalized tetrahedra, where the gluings are done by identifying faces, edges, and vertices. Self-gluing are allowed (that is, gluing a tetrahedra to itself), as are multiple gluings (for example, gluing two tetrahedra along more than one face). We refer the reader to [3] for a detailed description in the special case when only tetrahedra are used, known there as Δ complexes. If all the generalized tetrahedra are ideal (resp. semi ideal), then the generalized triangulation is called an *ideal* (resp. *semi ideal*) triangulation. If the quotient obtained is homeomorphic to a given manifold M it is said to be a *generalized triangulation of M* .

We refer the reader to, for example, [9, Section 2] for a detailed discussion of generalized tetrahedra. It is well known that a very large class of 3-manifolds admits generalized triangulations, including all compact 3-manifolds. We outline the proof here. Let N be a compact manifold and

$K_i \subset \partial N$ ($i = 1, \dots, n$) disjoint, closed, connected subsurface. By crushing each K_i to a point p_i , we obtain a 3-complex X . We can triangulate X so that each p_i is a vertex of the triangulation. Removing p_i we obtain a semi-ideal triangulation of $N \setminus (\cup_i K_i)$. We conclude that:

Lemma 5. *Let N be a compact manifold and let $K_i \subset \partial N$ be a closed, connected subsurface ($i = 1, \dots, n$), with $K_i \cap K_j = \emptyset$ for $i \neq j$. Then $N \setminus (\cup_i K_i)$ admits a generalized triangulation.*

In [4] J. Hempel defined the *distance* of a Heegaard splitting:

Definition 6. Let $V_1 \cup_\Sigma V_2$ be a Heegaard splitting and $n \geq 0$ an integer. We say that the distance of Σ is n , denoted by $d(\Sigma) = n$, if n is the smallest integer so that there exist meridian disks $D_1 \subset V_1$ and $D_2 \subset V_2$, and essential curves $\alpha_i \subset \Sigma$ ($i = 0, \dots, n$), so that $\alpha_0 = \partial D_1$, $\alpha_n = \partial D_2$, and for any i , $\alpha_{i-1} \cap \alpha_i = \emptyset$ ($1 \leq i \leq n$).

In [1], Casson and Gordon defined a Heegaard splitting $V_1 \cup_\Sigma V_2$ to be *weakly reducible* if there exist meridian disks $D_1 \subset V_1$ and $D_2 \subset V_2$ so that $D_1 \cap D_2 = \emptyset$, *strongly irreducible* otherwise. Note that Σ is weakly reducible if and only if $d(\Sigma) \leq 1$.

The following lemma is easy and well known (see, for example [19, Remark 2.6]):

Lemma 7. *Let $V_1 \cup_\Sigma V_2$ be a Heegaard splitting. Suppose that one of the following holds:*

- (1) *for $i = 1, 2$, there exists a properly embedded, non-boundary parallel annulus $A_i \subset V_i$, and there exist an essential curve $\alpha \subset \Sigma$ so that $\alpha \subset A_1 \cap A_2$, or:*
- (2) *there exist a meridian disk $D_1 \subset V_1$, a properly embedded, non-boundary parallel annulus $A_2 \subset V_2$, and an essential curve $\alpha \subset \Sigma$, so that $\alpha \cap D_1 = \emptyset$ and $\alpha \subset \partial A_2$.*

Then $d(\Sigma) \leq 2$.

3. COLORING Σ AND CONSTRUCTING THE PAIR OF PANTS X

Fix M as in the statement of Theorem 1 and let $V_1 \cup_\Sigma V_2$ be a Heegaard splitting for M with $g(\Sigma) \geq 76t + 26$.

Let \mathcal{T} be a generalized triangulation of M with t generalized tetrahedra. If Σ weakly reduces, then $d(\Sigma) \leq 1$; we assume from now on that Σ is strongly irreducible. Rubinstein [14] (see also Stocking [21], and Lackenby [8][9] when M is not closed) show that Σ is isotopic to an *almost normal* surface, that is, after isotopy the intersection of Σ with the generalized tetrahedra of \mathcal{T} consists of *normal faces*, of which there are two types: (1) normal disks (triangles or quadrilaterals), and: (2) an exceptional component, which is either an octagonal disk or an annulus obtained by tubing together 2 normal disks. At most one normal face of Σ is an exceptional component.

Let N be a regular neighborhood of $\mathcal{T}^{(1)}$, the 1-skeleton of \mathcal{T} . For each vertex $v \in \mathcal{T}^{(1)} \cap \Sigma$, let D_v be the component of $\Sigma \cap N$ containing v . Then D_v is a disk properly embedded in N , called the *vertex disk corresponding to v* . Let \widehat{F} be a normal face contained in a generalized tetrahedron T . Then $F = \widehat{F} \setminus \text{int}N$ is obtained from \widehat{F} by removing a neighborhood of the vertices of \widehat{F} . F is called a *truncated normal face*. For the remainder of this paper, by a *face* we mean a truncated normal face or a vertex disk.

Remark 8. The union of the boundaries of the faces forms a 3-valent graph in Σ .

Let $v, v' \in \mathcal{T}^{(1)}$ be two vertices, and $D_v, D_{v'}$ the corresponding vertex disks. Then D_v and $D_{v'}$ are called *I-equivalent* if v and v' are contained in the same edge $e \in \mathcal{T}^{(1)}$, and there exists an I -bundle $Q \subset N$, so that $\partial Q \setminus (D_v \cup D_{v'}) \subset \partial N$, $Q \cap \Sigma = D_v \cup D_{v'}$, and $D_v \cup D_{v'}$ is the associated ∂I -bundle. Note that D_v is *I-equivalent* to $D_{v'}$ if and only if v is adjacent to v' along an edge of $\mathcal{T}^{(1)}$.

Let F and F' be truncated normal faces. Then F and F' are called *I-equivalent* if F and F' are contained in the same generalized tetrahedron T , and there exists an I -bundle $Q \subset (T \setminus \text{int}N)$, so that $\partial Q \setminus (F \cup F') \subset \partial(T \setminus \text{int}N)$, $Q \cap \Sigma = F \cup F'$, and $F \cup F'$ is the associated ∂I -bundle. Note that F and F' are *I-equivalent* if and only if the corresponding normal faces are parallel, and there is no normal face in the region of parallelism between the two.

We emphasize that *I-equivalence* is not a transitive relation. The *I-equivalence* relation generates an equivalence relation on the faces. Its equivalence classes are called *I-equivalent families*.

Let \mathcal{F} be an *I-equivalent* family. Then the *I-equivalence* induces a linear ordering on the faces in \mathcal{F} , denoted in order F_1, \dots, F_n , so that F_i is *I-equivalent* to F_{i+1} ($i = 1, \dots, n-1$). This order is unique up-to reversing. We color the faces of \mathcal{F} as follows:

- (1) F_1 and F_n are colored dark red.
- (2) If $n \geq 3$, then F_2 and F_{n-1} are colored light red.
- (3) If $n \geq 5$, then F_3, \dots, F_{n-2} are colored blue and yellow alternately. Note that this leaves us the freedom to exchange the blue and yellow colors of the faces of \mathcal{F} .

We call a face *red* if it is either dark red or light red.

Remark 9. Let D_v be a red vertex disk. By construction, D_v is outermost or next to outermost along an edge of $\mathcal{T}^{(1)}$. Therefore all the truncated normal faces that intersect D_v are red as well.

A face is *I-equivalent* to two distinct faces if and only if it is colored blue, yellow, or light red. Let p be a point on such a face. Then p is on the boundary of two I -fibers, on the V_1 and V_2 sides. Denote the other endpoints of these fibers by p_1 and p_2 . By construction we see that the colors at p, p_1 and p_2 fulfill the conditions in Table 1.

p	p_1, p_2
blue	yellow or light red
yellow	blue or light red
light red	one is dark red and the other can be any color

TABLE 1.

Notation 10. Every light red face is *I-equivalent* to a dark red face on one side. On the other side it is *I-equivalent* to a face that may be blue, yellow, light red or dark red. This decomposes the set of light red points into four disjoint subsets. We label a light red point that co-bounds an I -fiber with a blue (resp. yellow) point by $\mathbf{lr}[\mathbf{b}]$ (resp. $\mathbf{lr}[\mathbf{y}]$).

Lemma 11. Let $f_{r,t}$ denote the number of the red truncated triangles and $f_{r,q}$ the number of the red truncated quadrilaterals. Then one of the following holds:

- (1) $f_{r,t} \leq 16t$ and $f_{r,q} \leq 4t + 4$.

(2) $f_{r,t} \leq 16t + 4$ and $f_{r,q} \leq 4t$.

Proof. A generalized tetrahedron not containing the exceptional component admits at most four I -equivalent families of truncated triangles and one I -equivalent family of truncated quadrilaterals. If there is an exceptional component, the generalized tetrahedron containing it admits at most five I -equivalent families of truncated triangles and one I -equivalent family of truncated quadrilaterals, or at most four I -equivalent families of truncated triangles and two I -equivalent families of truncated quadrilaterals. Each family contains at most four red faces. The lemma follows. \square

Let B (resp. Y , R) denote the union of the blue (resp. yellow, red) faces. By Remark 8, B , Y , R , and $B \cup Y$ are sub-surfaces of Σ .

Lemma 12. $\chi(B \cup Y) \leq -(108t + 38)$.

Proof. We first show that $\chi(R) \geq -(44t + 12)$; for that, we construct R by adding one face at a time, starting with the truncated exceptional component or the empty set if there is no such component. The truncated exceptional component has Euler characteristic 1 or 0, so worst case scenario is 0 (this includes the possibility that there is no exceptional component). The worst case scenario for attaching a truncated triangle is attaching it along three edges, in which case the Euler characteristic is reduced by 2. Similarly, attaching a truncated quadrilateral reduces the Euler characteristic by at most 3. By Remark 9, attaching a red vertex disk caps a red hole, and increases the Euler characteristic. Recall that $f_{r,t}$ and $f_{r,q}$ were defined and bounded in Lemma 11. Adding the contributions of the exceptional component (at worst 0), the triangles (at worst $-2f_{r,t}$), the quadrilaterals (at worst $-3f_{r,q}$), and ignoring the positive contribution of the vertex disks, Lemma 11 gives:

$$\begin{aligned} \chi(R) &\geq 0 - 2f_{r,t} - 3f_{r,q} \\ &\geq 0 - 2(16t) - 3(4t + 4) \\ &\geq -(44t + 12). \end{aligned}$$

Since $\Sigma = R \cup (B \cup Y)$ and $R \cap (B \cup Y)$ consists of circles, $\chi(B \cup Y) = \chi(\Sigma) - \chi(R)$. By assumption $g(\Sigma) \geq 76t + 26$, or equivalently $\chi(\Sigma) \leq 2 - 2(76t + 26)$. Hence:

$$\begin{aligned} \chi(B \cup Y) &= \chi(\Sigma) - \chi(R) \\ &\leq [2 - 2(76t + 26)] + [44t + 12] \\ &= -(108t + 38). \end{aligned}$$

\square

Denote the number of components by $|\cdot|$.

Lemma 13. $|\partial(B \cup Y)| \leq 44t + 14$.

Proof. By construction $\partial(B \cup Y) = \partial R$. Similar to the proof of the previous lemma, starting with the exceptional component or the empty set if there is no such component, we calculate the number of boundary components of R by adding one face at a time. The exceptional component has at most 2 boundary components, which is the worst case scenario (this includes the possibility that there is no exceptional component). The worst case scenario for attaching a truncated triangle is

attaching it along three edges, in which case the number of boundary components is increased by at most 2. Similarly, attaching a single truncated quadrilateral increases the number of boundary components by at most 3. By Remark 9, attaching a red vertex disk caps a red hole, and decreases the number of boundary components. Similar to the above, Lemma 11 gives:

$$\begin{aligned}
 |\partial(B \cup Y)| &= |\partial R| \\
 &\leq 2 + 2f_{r,t} + 3f_{r,q} \\
 &\leq 2 + 2(16t) + 3(4t + 4) \\
 &\leq 44t + 14.
 \end{aligned}$$

□

By Remark 8, $B \cap Y$ is a 1-manifold properly embedded in $B \cup Y$. Let $\Gamma \subset B \cup Y$ be the union of the arc components of $B \cap Y$. Endpoints of Γ are the points on Σ where red, blue, and yellow faces meet. Let \mathcal{V} be the set of vertices of red truncated normal faces. We subdivide \mathcal{V} into 3 disjoint sets as follows: \mathcal{V}_0 are vertices that are adjacent to at least two red faces; \mathcal{V}_+ are vertices that are adjacent to one red, one yellow, and one blue faces; \mathcal{V}_- are vertices that are adjacent to one red and two yellow faces, or one red and two blue faces. Remark 9 shows that every vertex where red, blue, and yellow faces meet belongs to \mathcal{V}_+ , *i.e.*, the red face is not a vertex disk. Thus \mathcal{V}_+ is exactly the set of endpoints of Γ .

We exchange the colors of the blue vertex disks with the colors of the yellow vertex disks; let R' , B' , Y' and Γ' be defined as above, using the new coloring. By Remark 9, \mathcal{V}_- is exactly the set of endpoints of Γ' (we emphasize that \mathcal{V}_- is the set of vertices defined above using the original coloring). Hence, by exchanging colors if necessary, we may assume that the number of endpoints of Γ is at most $\frac{1}{2}|\mathcal{V}|$. Since every arc of Γ has two distinct endpoints and Γ has at most $\frac{1}{2}|\mathcal{V}|$ endpoints, we get that $|\Gamma| \leq \frac{1}{4}|\mathcal{V}|$.

There are at most 16 vertices in \mathcal{V} from the truncated exceptional component, at most 6 from each truncated red triangle, and at most 8 from each truncated red quadrilateral. By Lemma 11 we get:

$$\begin{aligned}
 |\mathcal{V}| &\leq 16 + 6f_{r,t} + 8f_{r,q} \\
 &\leq 16 + 6(16t) + 8(4t + 4) \\
 &\leq 128t + 48.
 \end{aligned}$$

Hence:

$$|\Gamma| \leq \frac{1}{4}|\mathcal{V}| \leq 32t + 12.$$

Let F_1, \dots, F_k be the components of $B \cup Y$ cut open along Γ . Cutting increases the Euler characteristic by $|\Gamma|$ and increases the number of boundary components by at most $|\Gamma|$. Using Lemma 12 we get:

$$\begin{aligned}
 \sum_{i=1}^k \chi(F_i) &= \chi(\cup_{i=1}^k F_i) \\
 &= \chi(B \cup Y) + |\Gamma| \\
 &\leq -(108t + 38) + (32t + 12) \\
 &= -(76t + 26).
 \end{aligned}$$

And using Lemma 13 we get:

$$\begin{aligned}
 \sum_{i=1}^k |\partial F_i| &= |\partial \cup_{i=1}^k F_i| \\
 &\leq |\partial(B \cup Y)| + |\Gamma| \\
 &\leq (44t + 14) + (32t + 12) \\
 &= 76t + 26.
 \end{aligned}$$

Combining these inequalities we get:

$$(1) \quad \sum_{i=1}^k \chi(F_i) \leq -(\sum_{i=1}^k |\partial F_i|).$$

Proposition 14. *There exists a pair of pants $X \subset \Sigma$ with the following two properties:*

- (1) *Either every point of X is blue or every point of X is yellow (say the former).*
- (2) *The components of ∂X , denoted by α , β , and γ , are essential in Σ .*

Proof. By Inequality (1) above, for some i , $\chi(F_i) \leq -|\partial F_i|$; equivalently, $g(F_i) \geq 1$. Fix such i . By construction, $(B \cap Y) \cap \text{int} F_i$ consists of simple closed curves; see Figure 1. Let \mathcal{E} (resp. \mathcal{I}) denote the curves of $(B \cap Y) \cap \text{int} F_i$ that are essential (resp. inessential) in Σ .

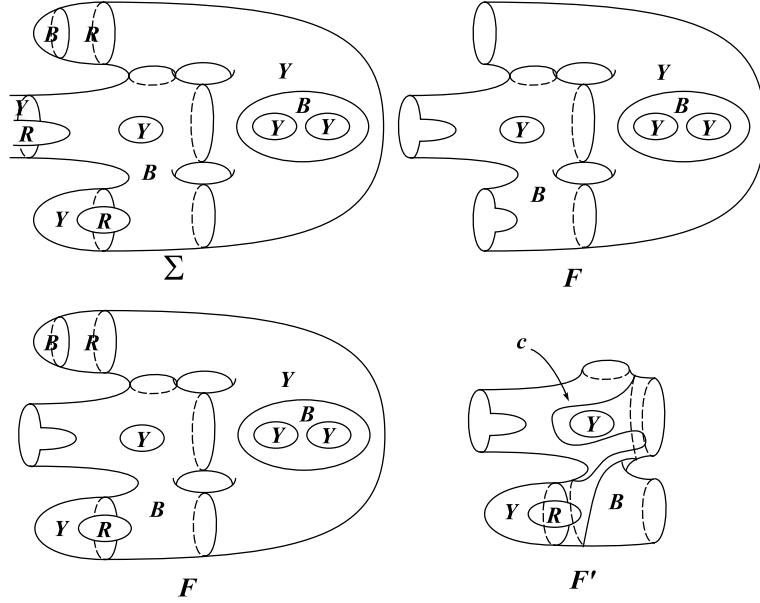


FIGURE 1.

Let Δ be the union of the components of $\text{cl}(\Sigma \setminus F_i)$ that are disks (possibly, $\Delta = \emptyset$). Let $F = F_i \cup \Delta$. By construction, every component of ∂F is essential in Σ (possibly, $\partial F = \emptyset$). Thus, a closed curve of F is essential in Σ if and only if it is essential in F . Since $g(F) \geq 1$, some component of F cut open along \mathcal{E} , denoted by F' , has $\chi(F') < 0$. Note that every curve of $\partial F'$ is essential in Σ . By construction, $(B \cap Y) \cap \text{int} F' \subset \mathcal{I}$. Let Δ' be the union of the disks bounded

by outermost curves of $\mathcal{I} \cap F'$ and the disks $\Delta \cap F'$. Note that $\Delta' \subset \text{int} F'$ consists of disks, and $F' \setminus \Delta'$ is entirely blue or yellow; in Figure 1, Δ' consists of two disks, one of each kind.

Assume first that $\partial F' \neq \emptyset$. Let $c \subset F'$ be a curve, parallel to a component of $\partial F'$, that decomposes F' as $A'' \cup_c F''$, where A'' is an annulus. By isotopy of c in F' we may assume that $\Delta' \subset A''$. We see that F'' is entirely blue or yellow, $\chi(F'') = \chi(F') < 0$, and $\partial F''$ is essential in Σ .

Next assume that $\partial F' = \emptyset$ (i.e., $F' = \Sigma$). Let c be a separating, essential curve in F' . By isotopy of c we may assume that Δ' is contained in one component of F' cut open along c . Let F'' be the other component. We conclude that in this case too, F'' is entirely blue or yellow, $\chi(F'') < 0$, and $\partial F''$ is essential in Σ .

Let α, β , and $\gamma \subset F''$ be three essential curves in F'' (and hence in Σ) that co-bound a pair of pants, denoted by X , in F'' . It is easy to see that X has the properties listed in Proposition 14. \square

Since X is entirely blue, X is on the boundary of I -bundles in V_i ($i = 1, 2$). The other component of the associated ∂I -bundle is a pair of pants denoted by X_i , with boundary components α_i (resp. β_i , and γ_i), parallel to α (resp. β , and γ). Since X is blue, every point of X_i is yellow or light red; in particular, $X \cap X_i = \emptyset$. The annuli extended from α (resp. β , and γ) in V_i are denoted by A_i (resp. B_i, C_i).

Lemma 15. *Let Q be a 3-dimensional I -bundle and let $A, A' \subset Q$ be immersed surfaces that are given as a union of fibers. Then the following hold:*

- (1) *A is embedded if and only if ∂A is embedded.*
- (2) *$A \cap A' = \emptyset$ if and only if $\partial A \cap \partial A' = \emptyset$.*

Proof. For (1), since A is a union of fibers, if A is not embedded then it intersects itself in fibers. The endpoints of these fibers show that ∂A is not embedded. The converse is trivial. The proof of (2) is similar and therefore omitted. \square

Since α is embedded so are α_1 and α_2 ; since α is blue, each point of α_1 and α_2 is yellow or light red. Thus $\alpha \cap \alpha_i = \emptyset$ and by Lemma 15(1), A_i is embedded ($i = 1, 2$). Similarly, B_1, B_2, C_1 , and C_2 are all embedded. The colors of α_i and X imply that $\alpha_i \cap X = \emptyset$ ($i = 1, 2$).

Lemma 16. *One of the following holds:*

- (1) *After renaming if necessary, either: $A_1 \subset V_1$ and $B_2 \subset V_2$ are not boundary parallel, and $A_2 \subset V_2, B_1 \subset V_1$, and $C_1 \subset V_1$ are boundary parallel.*
- (2) *$d(\Sigma) \leq 2$.*

Proof. We claim that one of A_i, B_i or C_i is not boundary parallel in V_i ($i = 1, 2$). Suppose, for a contradiction, that A_i, B_i, C_i are all boundary parallel. By Lemma 15(2), A_i, B_i , and C_i are mutually disjoint, and hence their regions of parallelisms are mutually disjoint or are nested. But the latter is impossible: suppose that the region of parallelism for $B_i \subset V_i$ is contained in the region of parallelism for $A_i \subset V_i$ (the other cases are symmetric). Then either X is contained in the annulus that A_i is parallel to, or X contains both curves of ∂A_i . In the former γ is inessential in Σ , contradicting Proposition 14(2), and in the latter $\alpha_i \subset X$, contradicting the comment preceding Lemma 16. Hence the regions of parallelisms are mutually disjoint, and the genus two surface $X \cup A_i \cup B_i \cup C_i \cup X_i$ is isotopic to Σ , contradicting the assumption that $g(\Sigma) \geq 76t + 26 > 2$.

Lemma 17. *Assume that Conclusion (1) of Lemma 16 holds. Then one of the following holds:*

-

Case One. α_1 can be isotoped to be disjoint from X_2 . Let \tilde{A}_2 , \tilde{B}_1 , and \tilde{C}_1 be as in Lemma 17. Let $T \subset \Sigma$ be the twice punctured torus $X \cup \tilde{B}_1 \cup \tilde{C}_1 \cup X_1$. Isotope α_1 so that $\alpha_1 \cap X_2 = \emptyset$. After this isotopy, $X_2 \cap (\alpha_1 \cup X) = \emptyset$. Hence either $X_2 \subset (X_1 \cup \tilde{B}_1 \cup \tilde{C}_1)$ or $X_2 \cap T = \emptyset$.

In the former case, $\alpha_2 \subset (X_1 \cup \tilde{B}_1 \cup \tilde{C}_1)$. Since α is isotopic to α_2 in Σ , α is isotopic into $X_1 \cup \tilde{B}_1 \cup \tilde{C}_1$. By Proposition 14(2) α is essential, and hence α is isotopic to a component of $\partial(X_1 \cup \tilde{B}_1 \cup \tilde{C}_1) = \alpha_1 \cup \beta \cup \gamma$; all contradict our assumptions.

Hence $X_2 \cap T = \emptyset$. Let $D_1 \subset V_1$ be a meridian disk obtained by compressing or boundary compressing A_1 . After a small isotopy we may assume that $\partial D_1 \cap \partial A_1 = \partial D_1 \cap (\alpha \cup \alpha_1) = \emptyset$, and hence either $\partial D_1 \subset T$ (hence $\partial D_1 \cap \beta_2 = \emptyset$) or $\partial D_1 \cap T = \emptyset$ (hence $\partial D_1 \cap \beta = \emptyset$). Using D_1 , B_2 , and β or β_2 , Lemma 7(2) shows that $d(\Sigma) \leq 2$, proving Theorem 1 in Case One.

Case Two. α_1 cannot be isotoped to be disjoint from X_2 . Since every point of α is blue, each point of α_1 is yellow or light red. Hence α_1 bounds I -bundles on both sides. Let $A_{1,2}$ be the annulus obtained by extending α_1 into V_2 , and denote $\partial A_{1,2} \setminus \alpha_1$ by $\alpha_{1,2}$; see Figure 4. Since α_1

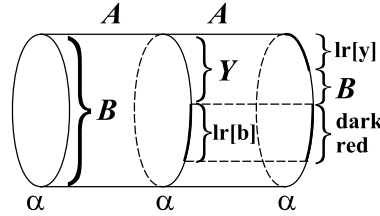


FIGURE 3.

is embedded, so is $\alpha_{1,2}$. Since every point of α_1 is yellow or light red and labeled $\mathbf{lr}[\mathbf{b}]$, every point of $\alpha_{1,2}$ is blue, light red and labeled $\mathbf{lr}[\mathbf{y}]$, or dark red (recall Table 1 and Notation 10). Thus $\alpha_1 \cap \alpha_{1,2} = \emptyset$. By Lemma 15(1) $A_{1,2}$ is embedded.

Since X_2 and X co-bound an I -bundle, every point of X_2 is yellow or light red and labeled $\mathbf{lr}[\mathbf{b}]$. Thus $\alpha_{1,2} \cap X_2 = \emptyset$. By assumption α_1 cannot be isotoped off X_2 . Hence α_1 is not isotopic to $\alpha_{1,2}$; this implies that $A_{1,2}$ is not boundary parallel. By assumption A_1 is not boundary parallel and α_1 is essential in Σ . Applying Lemma 7 (1) to A_1 , $A_{1,2}$, and α_1 we conclude that $d(\Sigma) \leq 2$, completing the proof of Theorem 1.

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